Exponential and Logarithmic Equations

To motivate you (at least a little bit), let's start with an example that involves making money. Say you have \$100, put it into a bank account that earns you 8% annual interest, and you want to know how many years it'll take before you've doubled your money. The equation that describes this situation is:

$$200 = 100(1.08)^t$$

You want to solve to find what t is. However, your variable, t, is in the exponent – in other words, this equation involves an exponential function. To solve it, the "regular" rules of algebra won't work – you have to use logarithms. Making lots and lots of money is just one reason to learn to solve equations involving exponentials and logarithms (another reason is that it's more fun than Disneyland).

In general, what logarithms and exponentials give us are two more tools in our toolbox. We can now "exponentiate" both sides or "logarithmify" both sides (note: logarithmify is not a real word). As with trig equations, we'll start with simple examples and get more complicated.

Let's say we have the equation $3^x = 9$ and want to know what x is. Of course, the answer is 2. However, we can use this simple example to illustrate our approach. One way of viewing this equation is as the right hand side of the definitive relationship:

$$y = \log_a x \text{ if and only if } x = a^y.$$

Therefore, $3^x = 9$ is true if and only if the alternative equation, $x = \log_3 9$ is true (the x in the equation $3^x = 9$ and the x in the equation $y = \log_a x$ of course refer to different things...we shouldn't be using the same letter, but there's only 26 to choose from so there will always be repeats floating around). And, what is $\log_3 9$? It's 2! So $x = 2$.

Another way to view this is as performing the same operation to both sides of the above equation, logarithmifying it. That is, we can start with our equation to solve, $3^x = 9$, and take the logarithm, base 3, of both sides:

$$\log_3(3^x) = \log_3 9.$$

The nice thing here is that $\log_3(3^x) = x$, by the inverse property of exponential and logarithmic functions. Therefore, the above equation simply reduces to $x = \log_3 9$, or $x = 2$. That kind of approach will work in general.

If we want to solve the equation $17^{x-2} = 8$, we can logarithmify both sides of the equation to get:

$$\log_{17} 17^{x-2} = \log_{17} 8,$$

which simplifies to

$$x - 2 = \log_{17} 8.$$

Adding 2 to both sides gives us that $x = \log_{17} 8 + 2$. Unlike the previous problem, we can't simplify this one any more without the help of a calculator since $\log_{17} 8$ is not a "nice" number (and if we used a calculator, our answer would not be exact any more, but just a decimal approximation of the actual answer). However, we could get an idea of the answer – since $17^0 = 1$ and $17^1 = 17$, $\log_{17} 8$ must be a number between 0 and 1. So our answer is likely about 2.5ish (it's actually about 2.73395...).

In the opposite direction, we can solve equations with logarithms in them by exponentiating both sides. Suppose we wanted to solve $\log_5(x + 1) = -3$. Then we could make both sides an exponent with base 5:

$$5^{\log_5(x+1)} = 5^{-3}$$

Again, the inverse properties of logarithms and exponents come to our rescue, since $5^{\log_5(x+1)} = x + 1$. Therefore, our equation reduces to $x + 1 = \frac{1}{125}$, so $x = -\frac{124}{125}$.

Use these tools to solve the following equations for x.

(1) $6^{2x+1} = 5$

(2) $\log_3(x-1) = 2$

(3) $\ln x = -1$

(4) $\log_3(\ln x) = 2$

Sometimes you may have to do some work first. The goal is similar to trig functions, to get your equation into the form "exponential = a number" or "logarithm = a number." For example, you could solve $\log_2 x + \log_2(x-3) - 4 = -2$ as follows:

$$\begin{aligned}
\log_2 x + \log_2(x-3) - 4 &= -2 \\
\log_2(x(x-3)) &= 2 \\
2^{\log_2(x(x-3))} &= 2^2 \\
x(x-3) &= 4 \\
x^2 - 3x - 4 &= 0 \\
(x-4)(x+1) &= 0 \\
x &= -1, 4
\end{aligned}$$

First we used one of the rules of logarithms to combine those two logarithms into a single log, did some algebra, exponentiated both sides, and then solved as we would a typical equation. But wait! Remember how we always say to check your answers? This is a good example why. Is $x = -1$ a solution to the original equation? That would mean $\log_2(-1) + \log_2(-4) - 4 = -2$. But $\log_2(-1)$ isn't even defined! So this can't be true...we picked up a "false" solution by accident. However, 4 is a solution, since $\log_2 4 = 2$, and $\log_2 1 = 0$, so $\log_2 4 + \log_2(4-3) - 4 = 2 - 0 - 4 = -2$.

Solve the following equations for x. Be sure to check and rule out any "false" solutions.

(5) $\log_2(x+1) - \log_2(x-1) - 3 = 0$

(6) $2 \cdot 4^{x+1} 4^{3x} = 10$

(7) $x - x \ln x = 0$

(8) $\log_3 x = 3 - \log_3 (x + 6)$

(9) $2^{x+1} + 2^x = 5$

Now, in all of the examples so far it's been relatively obvious what base to use as our logarithm or what base to use in our exponential. What if it's not so obvious? Take, for example, the equation $2^x = 3^{x-1}$. Should we use a logarithm base 2 or a logarithm base 3? The truth is, it doesn't actually matter. We can do it either way.

$$
\begin{aligned}
2^x &= 3^{x-1} \\
\log_2(2^x) &= \log_2(3^{x-1}) \\
x &= \log_2(3^{x-1}) \\
x &= (x-1)\log_2(3) \\
x &= x\log_2(3) - \log_2(3) \\
x - x\log_2(3) &= -\log_2(3) \\
x(1 - \log_2(3)) &= -\log_2(3) \\
x &= \frac{-\log_2(3)}{1 - \log_2(3)}
\end{aligned}
$$

Now let's solve it the other way:

$$\begin{aligned}
2^x &= 3^{x-1} \\
\log_3(2^x) &= \log_3(3^{x-1}) \\
\log_3(2^x) &= x-1 \\
x\log_3(2) &= x-1 \\
x\log_3(2) - x &= -1 \\
x(\log_3(2)-1) &= -1 \\
x &= \frac{-1}{\log_3(2)-1}
\end{aligned}$$

Either way is valid. But wait..., aren't those different answers? How can they both be right? As a matter of fact, the answers may look different, but they're exactly the same! Try plugging them into your calculator – you'll get about 2.7 in either case. If you still don't believe it (good for you, ask for proof!), we can prove it using the change of base formula for logarithms. Remember, $\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$, so $\log_2(3) = \frac{\log_3(3)}{\log_3(2)} = \frac{1}{\log_3(2)}$.

$$\frac{-\log_2(3)}{1-\log_2(3)} = \frac{\frac{-1}{\log_3(2)}}{1-\frac{1}{\log_3(2)}} = \frac{\frac{-1}{\log_3(2)}}{\frac{\log_3(2)-1}{\log_3(2)}} = \frac{-1}{\log_3(2)-1}$$

In fact, we could have used a logarithm with any base in order to solve this problem, we didn't even have to restrict ourselves to base 2 or base 3 (although those two make it simplest). Let's explore that idea further with one more example, returning to our original goal of solving $200 = 100(1.08)^t$. This equation involves an exponential with a base of 1.08. That's not a base you're going to use much (or find on many calculators, if you decide to use one in the future). We'll solve it using the natural logarithm. We'll start by taking the logarithm of both sides, and then apply some rules of logarithms to simplify.

$$\begin{aligned}
200 &= 100(1.08)^t \\
\ln 200 &= \ln(100(1.08)^t) \\
\ln 200 &= \ln 100 + \ln(1.08)^t \\
\ln 200 &= \ln 100 + t\ln(1.08) \\
\ln 200 - \ln 100 &= t\ln(1.08) \\
\frac{\ln 2}{\ln 1.08} &= t
\end{aligned}$$

There are a few steps combined into one toward the end – make sure you can figure them all out, it's good practice. Therefore, t is $\frac{\ln 2}{\ln 1.08}$, or if you had a calculator around, $t \approx 9$. It would take you 9 years to double your money.

(10) Solve $7^a = 4^{a-1}$ for a.

(11) Solve $2^{2x}(3^{x+1}) = 5^x$ for x.

(12) If you invest \$5000 into an account with 5% annual interest, how long will it take for your account to triple in value?